Martin Li

Recall:

$f : \mathbb{R} \to \mathbb{R}$

$y = f(x)$

Single variable calculus.

$\lim_{x \to c} f(x)$, $f'(x)$, $\int_a^b f(x)\,dx$

$\boxed{\begin{array}{l} f : \mathbb{R} \longrightarrow \mathbb{R}^m \quad (\text{curves in } \mathbb{R}^m) \\ f(t) = (x_1(t), \ldots, x_m(t)) \end{array}}$ today

$F : \mathbb{R}^n \longrightarrow \mathbb{R}$ (multivariable function)

$F(x_1, \ldots, x_n) \in \mathbb{R}$

$F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (most general).

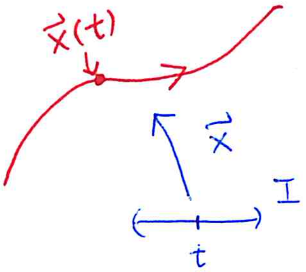
Curves in $\mathbb{R}^m$

Def$^{\underline{n}}$: Let $I \subset \mathbb{R}$ be an interval.

A curve in $\mathbb{R}^m$ is a (vector-valued) function

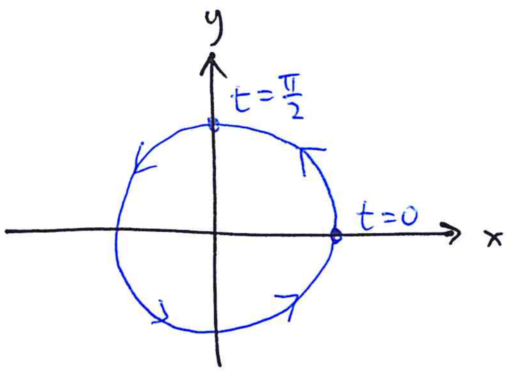
$$\vec{x} : I \longrightarrow \mathbb{R}^m$$

$$\vec{x}(t) = (\underbrace{x_1(t), \ldots, x_m(t)}_{\text{component functions.}}) \qquad t \in I.$$

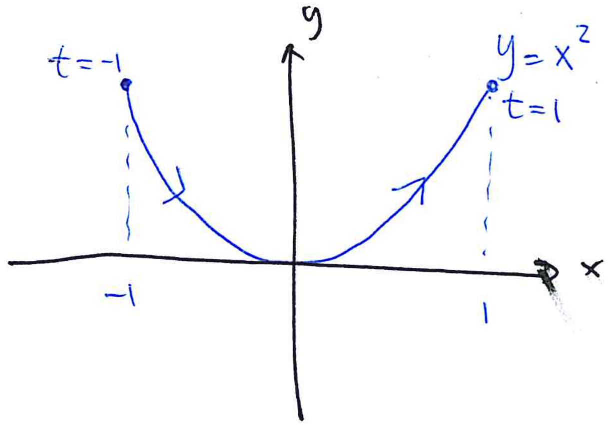


Examples:

(1) $\vec{x}(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$

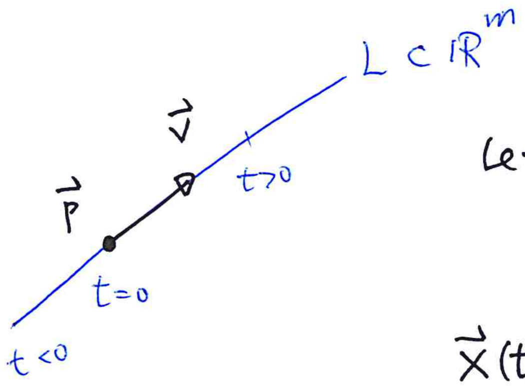


(2) $\vec{x}(t) = (t, t^2)$, $t \in [-1, 1]$.



parabola .

Remark: A curve is the geometric object together with a "parametrization".



$L = \{ \vec{p} + t\vec{v} \mid t \in \mathbb{R} \}$. parametric form .

Let $\vec{p} = (p_1, p_2, p_3)$

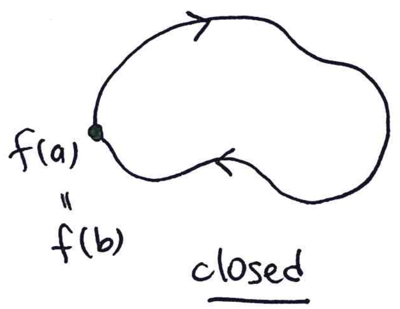
$\vec{v} = (v_1, v_2, v_3)$

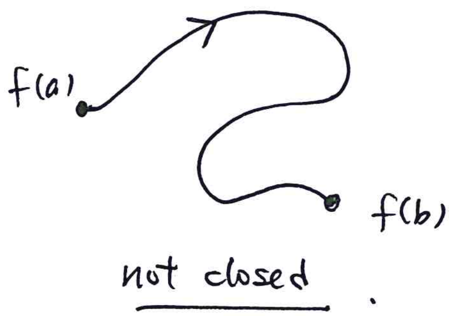
$$\vec{x}(t) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3) \qquad t \in \mathbb{R}.$$

$$\vec{y}(t) = (p_1 + 2tv_1, p_2 + 2tv_2, p_3 + 2tv_3) \quad , t \in \mathbb{R}.$$

different parametrization of the same line .

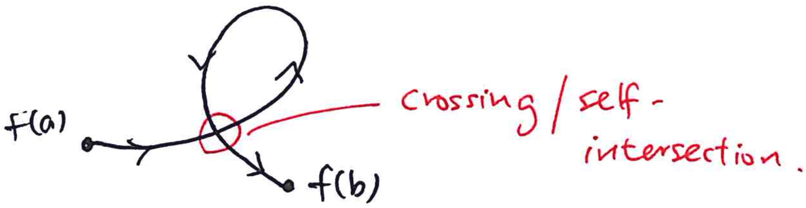
Closed Curves : Def$^{\underline{n}}$: $f : [a,b] \to \mathbb{R}^m$ is closed if $f(a) = f(b)$





Simple curves: **Def$^{\underline{n}}$** : $f:[a,b]\longrightarrow \mathbb{R}^m$ is **simple** if f is 1-1 on (a,b).

simple



not simple

Analytic Properties

Let $\vec{x}: I \longrightarrow \mathbb{R}^m$, $t_0 \in I$.

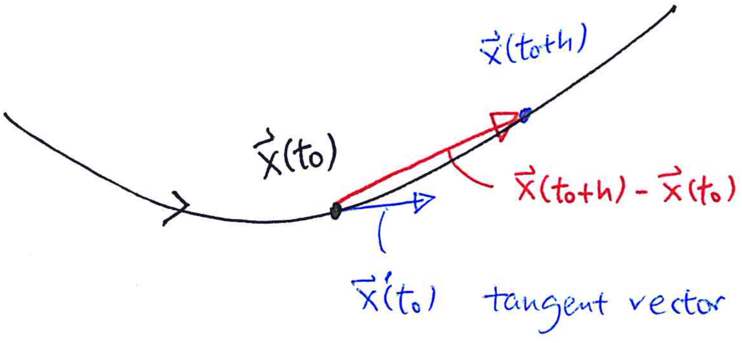
limit: $\displaystyle\lim_{t\to t_0} \vec{x}(t)$ **derivative**: $\vec{x}\,'(t)$

If $\vec{x}(t) = (x_1(t), \ldots, x_m(t))$, then

$$\lim_{t\to t_0}\vec{x}(t) = \underbrace{\left(\lim_{t\to t_0} x_1(t), \ldots, \lim_{t\to t_0} x_m(t)\right)}_{\text{if they exists}}$$

$$\vec{x}\,'(t_0) = \left(\overbrace{x_1'(t_0), \ldots, x_m'(t_0)} \right) = \lim_{h\to 0}\frac{\vec{x}(t_0+h)-\vec{x}(t_0)}{h}$$

Picture:



$\|\vec{x}\,'(t_0)\|$: speed at t_0

Physics:

$\vec{x}\,'(t_0)$: velocity.

$\vec{x}\,''(t_0)$: acceleration

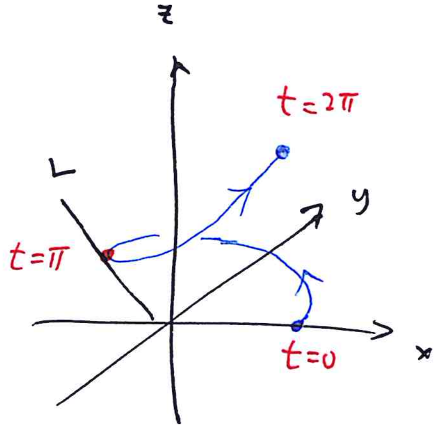
$$\boxed{F = ma}$$
$$\Updownarrow$$
$$\boxed{\vec{F} = m\vec{x}\,''}$$

Arc length: $\vec{x} : [a,b] \to \mathbb{R}^m$

$$\text{arc length} := \int_a^b \|\vec{x}'(t)\|\, dt.$$

Example (Helix)

$$\vec{x}(t) = (\cos t, \sin t, t), \quad t \in [0, 2\pi]$$



Find (a) the tangent line at $t=\pi$

(b) arc length of the helix.

Sol: (a) At $t=\pi$,

$$\vec{p} = \vec{x}(\pi) = (-1, 0, \pi)$$

$$\vec{x}'(t) = (-\sin t, \cos t, 1)$$

$$\vec{x}'(\pi) = (0, -1, 1) = \vec{v}$$

$$L = \{ (-1, 0, \pi) + t\,(0, -1, 1) \mid t \in \mathbb{R} \}$$ tangent line at $t=\pi$.

(b) $$\|\vec{x}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

$$\text{arc length} = \int_0^{2\pi} \|\vec{x}'(t)\| dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$ ※

Diff. Rules:

(1) $[\vec{x}(t) \pm \vec{y}(t)]' = \vec{x}'(t) \pm \vec{y}'(t)$

(2) $[c\,\vec{x}(t)]' = c\,\vec{x}'(t)$

(3) $[f(t)\,\vec{x}(t)]' = f'(t)\,\vec{x}(t) + f(t)\,\vec{x}'(t)$

(4) $[\vec{x}(t) \cdot \vec{y}(t)]' = \vec{x}'(t) \cdot \vec{y}(t) + \vec{x}(t) \cdot \vec{y}'(t)$

$m=3$ (5) $[\vec{x}(t) \times \vec{y}(t)]' = \vec{x}'(t) \times \vec{y}(t) + \vec{x}(t) \times \vec{y}'(t)$

$\Big\}$ product rules.

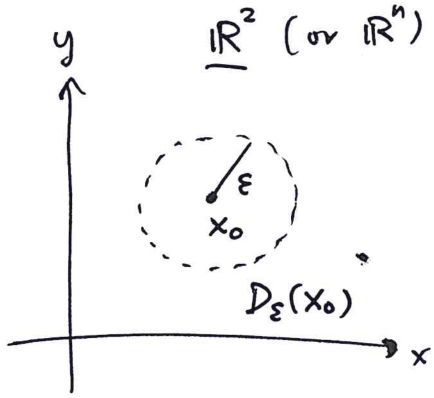
Q: Prove these!

Last time ... $\vec{x} : I \subset \mathbb{R} \longrightarrow \mathbb{R}^m$

$$\vec{x}(t) = (x_1(t), \dots, x_m(t))$$

Basic Point Set Topology ($\Omega \subset \mathbb{R}^n$)

In $\mathbb{R}^n$, denote.

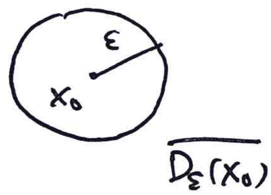


$$D_\varepsilon(x_0) := \{ x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon \}$$

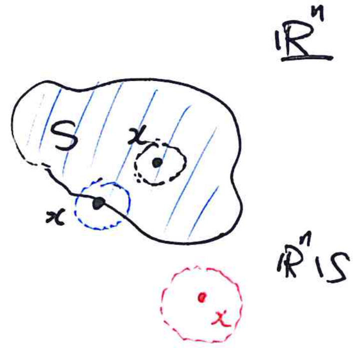
open ball with radius ε and center at x_0

$$\overline{D_\varepsilon(x_0)} := \{ x \in \mathbb{R}^n \mid \|x - x_0\| \leq \varepsilon \}$$

closed ball with radius ε, center at x_0



· If $S \subset \mathbb{R}^n$, we can classify $x \in \mathbb{R}^n$ into one of the following types

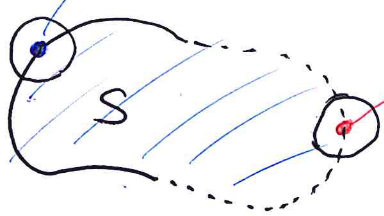


(1) interior points : $D_\varepsilon(x) \subset S$ for some $\varepsilon > 0$

(2) exterior points : $D_\varepsilon(x) \subset \mathbb{R}^n \setminus S$ for some $\varepsilon > 0$

(3) boundary points : $D_\varepsilon(x) \cap S \neq \emptyset$, $D_\varepsilon(x) \cap (\mathbb{R}^n \setminus S) \neq \emptyset$ for all $\varepsilon > 0$

Remark: We may have boundary points lying in S or not in S.



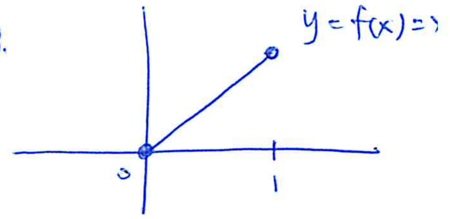
Denote: $\partial S := \{ x \in \mathbb{R}^n \mid x \text{ is a boundary point of } S \}$.

Generalize: (a, b) open interval $\rightarrow$ derivative.

$[a, b]$ closed interval $\rightarrow$ optimization

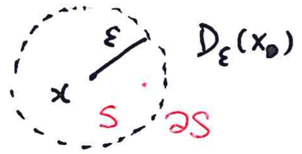
min/max $f(x)$

e.g.



Def$^{\underline{n}}$: Let $S \subset \mathbb{R}^n$.

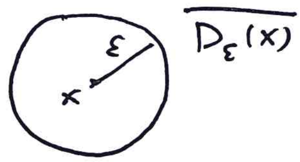
(i) S is open: $\partial S \subset \mathbb{R}^n \setminus S$



i.e. $\forall\, x_0 \in S$, $\exists\, \varepsilon > 0$ s.t. $D_\varepsilon(x_0) \subset S$.

[Remark: ε depends on the point x_0]

(ii) S is closed: $\partial S \subset S$.



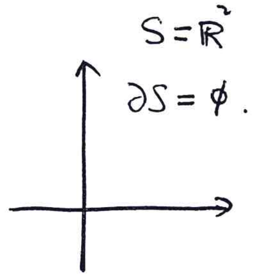
Ex: S is closed $\Longleftrightarrow$ $\mathbb{R}^n \setminus S$ is open.

Remark: Some sets are not open nor closed.

e.g. $(a, b]$

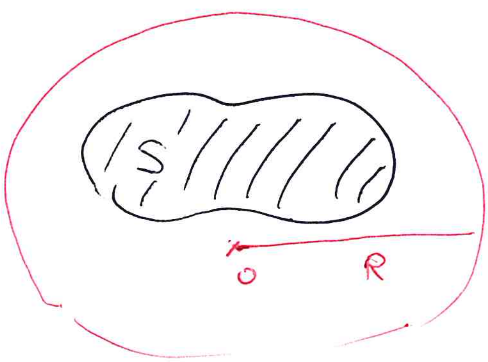
Some sets are both open and closed in $\mathbb{R}^n$.

only two cases: $S = \phi$ and $\mathbb{R}^n$



Def$^{\underline{n}}$: (1) $S \subset \mathbb{R}^n$ is compact if S is closed and bounded.

$\Leftarrow$ $\exists\, R > 0$ large s.t. $S \subset D_R(0)$.

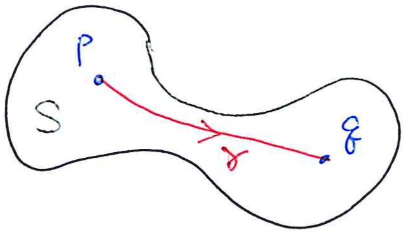


"Compact"

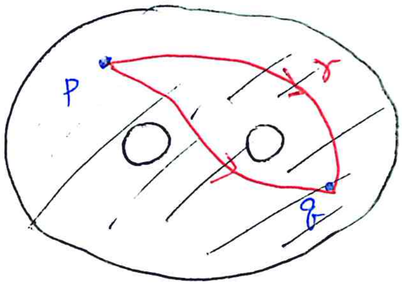
$\Updownarrow$

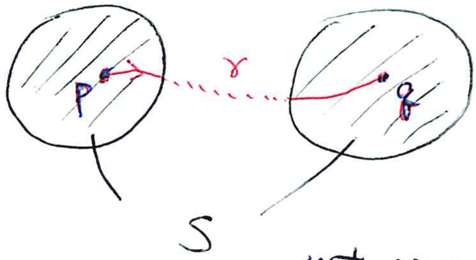
"finiteness"

(2) $S \subset \mathbb{R}^n$ is connected if any two points $p, q \in S$ can be joined by a continuous curve γ inside S.



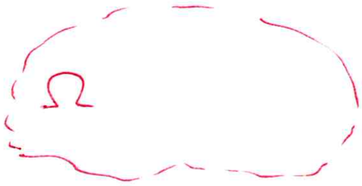
connected





not connected

(3) $\Omega \subset \mathbb{R}^n$ is domain if it is open & connected.



Continuity: $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, Ω : domain

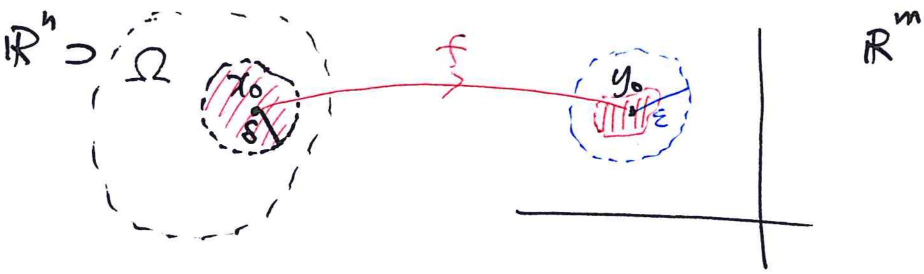
f is continuous $\iff$ (i) $\forall x_0 \in \Omega$, $\lim_{x \to x_0} f(x) = f(x_0)$

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\| f(x) - f(x_0) \| < \varepsilon$ if $\| x - x_0 \| < \delta$.

$\iff$ (ii) $\forall y_0 \in \mathbb{R}^m$, $\forall \varepsilon > 0$,

$f^{-1}(D_\varepsilon(y_0))$ is open.



Multivariable Function

$$F : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m \qquad \Omega : \text{domain}$$

$$F(x_1, \dots, x_n) = \underbrace{(f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))}_{\text{component functions of } F} .$$

Fact: $\lim\limits_{x \to x_0} F(x) = \left(\lim\limits_{x \to x_0} f_1(x), \dots, \lim\limits_{x \to x_0} f_m(x) \right)$

$\Rightarrow$ We just need to understand the case $m = 1$

$$f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \left(\begin{array}{l} \text{real-valued} \\ n \text{ indep. variables} \end{array} \right)$$

$$f = f(x_1, \dots, x_n)$$

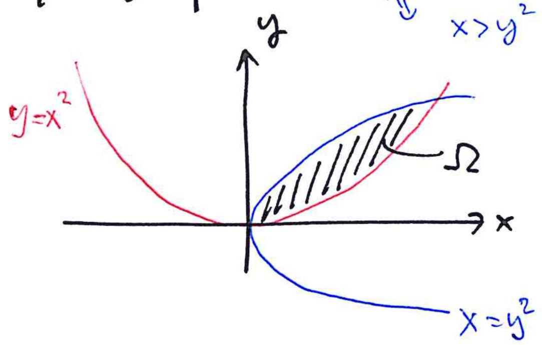
Example: $f(x,y) = x^2 - y^2 \quad , \quad \Omega = \mathbb{R}^2$

$$F(x,y) = \left(\ln(x - y^2), \ln(y - x^2) \right)$$

$$F : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

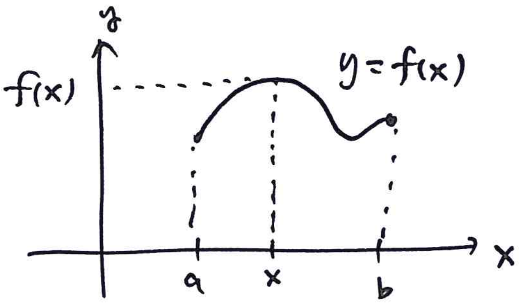
$$\Omega = \{ (x,y) \mid \underset{\Updownarrow \; x > y^2}{x - y^2 > 0} \text{ and } \underset{\Updownarrow \; y > x^2}{y - x^2 > 0} \}$$

$\ln x \ , \ x > 0$

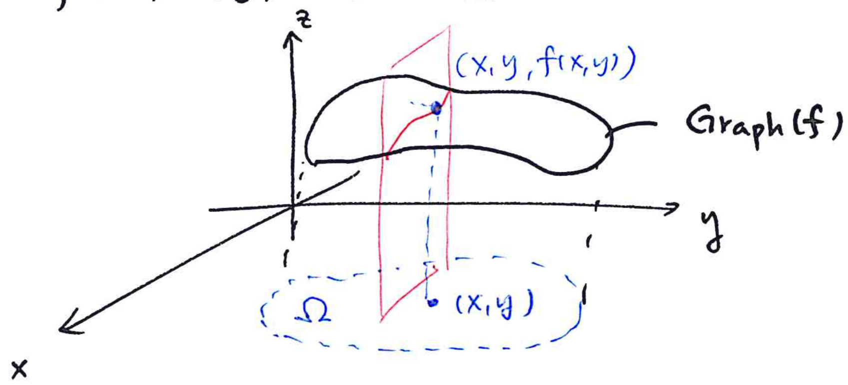


Q: How to understand $f(x_1, \dots, x_n)$?

Graph of f: $\quad f : [a,b] \longrightarrow \mathbb{R}$



$$f = f(x,y) : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$



<u>Defⁿ</u>: If $f(x_1,\dots,x_n) : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, then

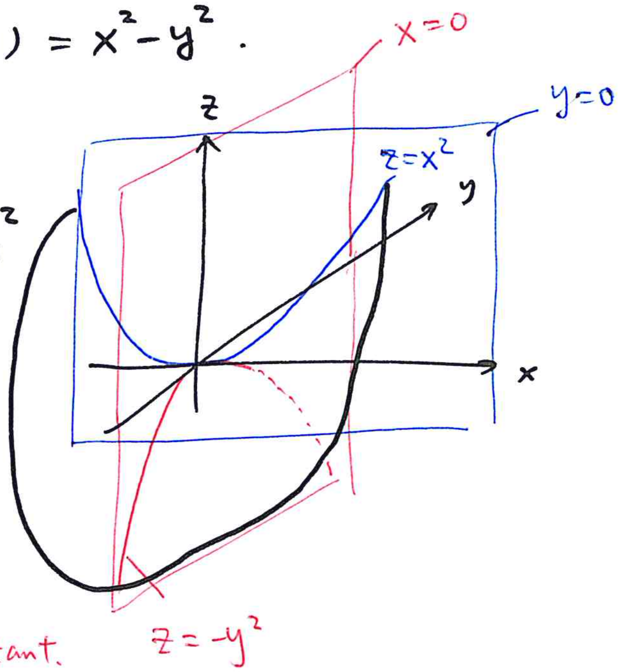
$$\mathrm{Graph}(f) := \{ (x_1,\dots,x_n, f(x_1,\dots,x_n)) \mid (x_1,\dots,x_n) \in \Omega \}$$

$$= \{ (x_1,\dots,x_n,u) \mid u = f(x_1,\dots,x_n),\ (x_1,\dots,x_n)\in\Omega \} \subset \mathbb{R}^{n+1}$$

<u>Example</u>: Sketch the graph of $f(x,y) = x^2 - y^2$.

<u>Idea of "slicing"</u>:

ertical
$$\begin{cases} \text{Fix } x = 0, & \cancel{f(x,} \ f(0,y) = -y^2 \\ \text{Fix } y = 0, & f(x,0) = x^2 \end{cases}$$



<u>Horizontal slicing</u>:

Slice by horizontal planes: $z = c$

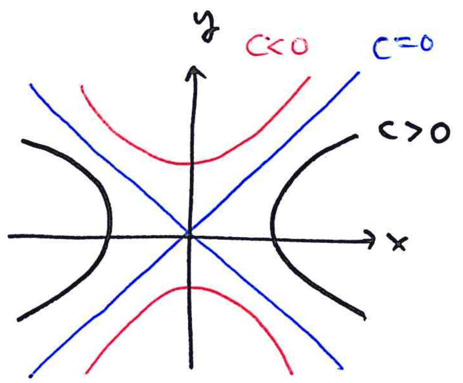
Constant.

$$\begin{cases} z = f(x,y) \\ z = c \end{cases} \Rightarrow \boxed{f(x,y) = c}$$ contour / level curve.

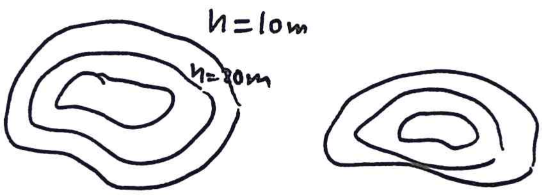
$x^2 - y^2 = 0 \Rightarrow y = \pm x$

$x^2 - y^2 = c > 0 \Rightarrow$ hyperbola!

$x^2 - y^2 = c < 0 \Rightarrow$



map: (geography)



Example: $f(x,y,z) = x^2+y^2-z^2$. $\quad f : \underset{\Omega}{\overset{\shortparallel}{\mathbb{R}^3}} \longrightarrow \mathbb{R}$.

- we cannot draw the graph (need 4 dim).
- slice (horizontal)

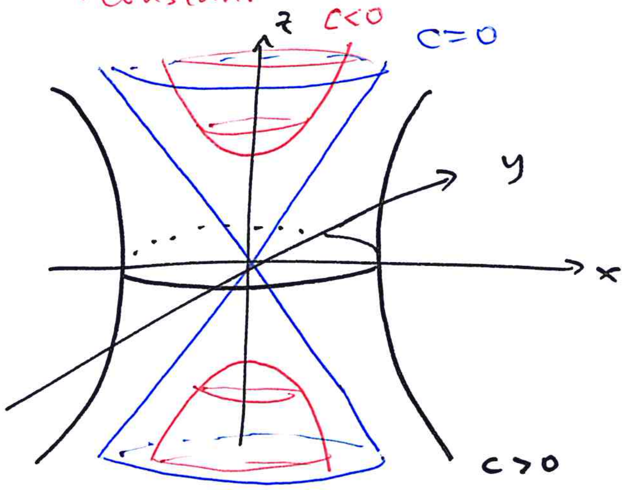
$$f(x,y,z) = c \qquad \text{2D surface in } \mathbb{R}^3$$

constant

$x^2+y^2-z^2 = 0$

$x^2+y^2-z^2 = c > 0$

$x^2+y^2-z^2 = c < 0$



Generalize: $f(x_1, \dots, x_n)$

level set: $L_c := \{ (x_1, \dots, x_n) \in \Omega \subset \mathbb{R}^n \mid f(x_1, \dots, x_n) = c \}$